ISOMETRIC EMBEDDINGS OF TEICHMÜLLER SPACES ARE COVERING CONSTRUCTIONS

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Abstract. Let $h : \Sigma' \to \Sigma$ be a branched covering of topological surfaces. By pulling back complex structures, h induces a holomorphic isometric embedding of Teichmüller spaces $\mathcal{T}(\Sigma) \hookrightarrow \mathcal{T}(\Sigma')$. We show that for $\dim \mathcal{T}(\Sigma) \geq 2$, all isometric embeddings arise from branched coverings. This generalizes a theorem of Royden [Roy71]. As a consequence, we obtain that totally geodesic submanifolds of $\mathcal{T}(\Sigma)$, which are isometric to some Teichmüller space, are covering constructions. Another consequence is the classification of local isometric embeddings of moduli spaces of Riemann surfaces.

1. Introduction

Let $\mathcal{T}_{g,n} = \mathcal{T}(\Sigma)$ be the Teichmüller space of an n-marked topological surface Σ of genus g, where we assume $3g - 3 + n > 0$. A celebrated theorem of Royden [Roy71], combined with results by Earle-Kra [EK74], shows that for $2g + n \geq 5$ the mapping class group, the biholomorphism group of $\mathcal{T}_{g,n}$ and the (orientation-preserving) isometry group for the Teichmüller metric are isomorphic. In particular all isometries for the Teichmüller metric are induced by a diffeomorphism of topological surfaces. In this paper we study a generalization of Royden's theorem by relaxing isometries to isometric embeddings, namely, distance-preserving maps for the Teichmüller metric. One source of isometric embeddings is from *covering constructions*, which can be described as follows. A point in $(X, \phi) \in \mathcal{T}(\Sigma)$ is defined by a marked Riemann surface $\phi : \Sigma \to X$, where ϕ is a homeomorphism from a fixed topological surface Σ to a Riemann surface X. Suppose that

$$h : \Sigma' \to \Sigma$$

is a branched covering of topological surfaces. Then there exists a unique marked Riemann surface $\phi_h : \Sigma' \to X_h$ such that the composition

$$\phi \circ h \circ (\phi_h)^{-1} : X_h \to X$$

is holomorphic. We call a map

$$f : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$$

a *covering construction* if there exists a branched covering $h : \Sigma' \to \Sigma$ such that for all $(X, \phi) \in \mathcal{T}(\Sigma)$ we have

$$f(X, \phi : \Sigma \to X) = (X_h, \phi_h : \Sigma' \to X_h).$$

We will discuss covering constructions in more detail in Section 2 and in the Appendix A.

Our main result is that, except for finitely many exceptions of g and n, covering constructions are the only examples of isometric embeddings.

Theorem 1.1 (**Isometric embeddings are geometric**). *Let $\mathcal{T}_{g,n}$ and $\mathcal{T}_{g',n'}$ be Teichmüller spaces and suppose $2g + n \geq 5$. Then any isometric embedding $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ is a covering construction, up to pre- or post-composition with an orientation-reversing mapping class.*

In [Ant17], Antonakoudis showed that isometric embeddings are either holomorphic or antiholomorphic. Thus, by possibly composing with an orientation-reversing mapping class, we can focus on holomorphic, isometric embeddings. We also have a classification in the exceptional cases $2g + n \leq 4$, although the classification is more subtle. The only four possibilities for g and n are $(g, n) = (0, 4), (1, 1), (1, 2), (2, 0)$. The first two are 1-dimensional Teichmüller spaces, the remaining two are of dimension 2 and 3, respectively.

Corollary 1.2 (**Low complexity cases**). *Let $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ be a holomorphic, isometric embedding of Teichmüller spaces. Assume $2g + n \leq 4$.*

(1) *If $\dim \mathcal{T}_{g,n} = 1$, then f is a Teichmüller disc.*
(2) *If $(g, n) = (1, 2)$, then $\iota : \mathcal{T}_{1,2} \simeq \mathcal{T}_{0,5}$ via the hyperelliptic involution and*

$$f = h \circ \iota$$

for some covering construction $h : \mathcal{T}_{0,5} \to \mathcal{T}_{g',n'}$.
(3) *If $(g, n) = (2, 0)$, then $\iota : \mathcal{T}_{2,0} \simeq \mathcal{T}_{0,6}$ via the hyperelliptic involution and*

$$f = h \circ \iota$$

for some covering construction $h : \mathcal{T}_{0,6} \to \mathcal{T}_{g',n'}$.

Proof. The case of 1-dimensional Teichmüller spaces was already considered by Antonakoudis [Ant17, Thm 1.1]. For the remaining Teichmüller spaces we can compose any isometric embedding with an isomorphism to either $\mathcal{T}_{0,5}$ or $\mathcal{T}_{0,6}$, which both satisfy the assumptions of Theorem 1.1. □

1.1. **Motivation via Teichmüller dynamics.** The image of an isometric embedding is an example of totally geodesic complex submanifold, i.e., a complex submanifold $M \subseteq \mathcal{T}_{g,n}$ such that for any two points $x, y \in M$ the unique geodesic through x and y lies in M. Wright [Wri20] proved that totally geodesic submanifolds of dimension at least 2 in Teichmüller space project to subvarieties of $\mathcal{M}_{g,n}$ and that there are only finitely many totally geodesic subvarieties of dimension at least 2 in $\mathcal{M}_{g,n}$ for fixed g, n. It was originally conjectured by Mirzakhani that every totally geodesic submanifold of Teichmüller space is the image of a covering construction, but recently in [MMW17, EMMW20] McMullen, Mukamel, Wright and Eskin discovered examples of totally geodesic submanifolds which are not isometric to any Teichmüller space. Our result about isometric embeddings can be rephrased in this setting as the following.

Corollary 1.3. *Let $M \subseteq \mathcal{T}_{g',n'}$ be a totally geodesic submanifold with $\dim(M) \geq 2$. If M is isometric (as metric space) to some Teichmüller space with the Teichmüller metric, then M is the image of a holomorphic covering construction $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ for some Teichmüller space $\mathcal{T}_{g,n}$.*

Proof. Let $\mathcal{T}_{g,n}$ be a Teichmüller space which is isometric to M. By composing an isometry between $\mathcal{T}_{g,n}$ and M with the embedding of M in $\mathcal{T}_{g',n'}$, we obtain an isometric embedding $f : \mathcal{T}_{g,n} \hookrightarrow \mathcal{T}_{g',n'}$ with $f(\mathcal{T}_{g,n}) = M$. After possibly changing the orientation we can assume f is holomorphic. If $2g + n \geq 5$, then f is a covering construction by Theorem 1.1. In the remaining case $2g+n \leq 4$ we conclude from $\dim \mathcal{T}_{g,n} = \dim M \geq 2$, that either $(g, n) = (1, 2)$ or $(g, n) = (2, 0)$. We proceed as in the proof of Corollary 1.2 and use the isometries $\mathcal{T}_{1,2} \simeq \mathcal{T}_{0,5}$ and $\mathcal{T}_{2,0} \simeq \mathcal{T}_{0,6}$ to reduce to the previous case $2g + n \geq 5$. □

1.2. **Rigidity of isometric embeddings.** Covering constructions descend to algebraic maps of finite coverings of the moduli space of curves and thus we obtain a rigidity result for isometric embeddings from Theorem 1.1.

Corollary 1.4. *Let $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ be an isometric embedding and assume* $\dim \mathcal{T}_{g,n} \geq 2$. *Then there exists a finite cover $\widetilde{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,n}$ such that f descends to an algebraic, local isometry $\widetilde{f} : \widetilde{M}_{g,n} \to \widetilde{M}_{g',n'}$ of corresponding coarse moduli spaces. Furthermore, assume that $f : \widetilde{\mathcal{M}}_{g,n} \to \widetilde{\mathcal{M}}_{g',n'}$ is holomorphic and a local isometric embedding of orbifolds, where*

$$\widetilde{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,n},\, \widetilde{\mathcal{M}}_{g',n'} \to \mathcal{M}_{g',n'}$$

are finite orbifold covering maps and $\dim \mathcal{M}_{g,n} \geq 2$. *Then, f is induced by a covering construction $\widetilde{f} : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$. Furthermore, the set of holomorphic, local isometric embeddings $f : \widetilde{\mathcal{M}}_{g,n} \to \widetilde{\mathcal{M}}_{g',n'}$ is finite.*

We will give the precise definition of local isometries for coarse moduli spaces and orbifolds, as well as the proof of Corollary 1.4, in Section 5.1

Related work. In a different direction, Royden's theorem can be generalized by considering isometric submersions. An *isometric submersion* is a holomorphic map $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ such that the induced map $df : T\mathcal{T}_{g,n} \to T\mathcal{T}_{g',n'}$ on tangent spaces maps the unit ball onto the unit ball. In [GG22] Gekhtman and Greenfield show that, if $g' \geq 1, 2g' + n' \geq 5$, every holomorphic isometric submersion $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g,',n'}$ for the Teichmüller metric is a forgetful map $\mathcal{T}_{g,n} \to \mathcal{T}_{g,m}, n \geq m$, which forgets some of the marked points.

We also note that there are examples of holomorphic maps of moduli spaces that are neither isometric embeddings nor isometric submersions, for example in [ALS08] Aramayona, Leininger and Souto constructed an example by composing an isometric covering construction with a forgetful map, i.e., an isometric embedding followed by an isometric submersion.

1.3. **Outline of the proof of Theorem 1.1.** By [Ant17] every orientation-preserving isometric embedding $f : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$ is holomorphic. Thus from now on we assume f is holomorphic. To show that f is a covering construction one needs a way to build a branched covering from f. At each point $X \in \mathcal{T}(\Sigma)$ we can identify the cotangent space $T^*_X\mathcal{T}(\Sigma)$ with the space of integrable quadratic differentials $Q(X)$, and the map f induces a map $f^* : Q(Y) \to Q(X)$ between cotangent spaces, where $Y = f(X)$. If f were a biholomorphism, as in the case of Royden's theorem, then $f^* : Q(Y) \to Q(X)$ is a $\mathbb{C}$-linear isometry. Royden [Roy71] then showed that every $\mathbb{C}$-linear isometry $Q(Y) \to Q(X)$ arises from an isomorphism $X \to Y$. In [Mar03] Marković gives a new proof of Royden's theorem and shows more generally that every $\mathbb{C}$-linear isometric embedding $Q(Y) \to Q(X)$ of spaces of quadratic differentials is induced by a branched covering $X \to Y$. It follows that every holomorphic map $f : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$, such that f^* is an isometric embedding, is a forgetful map, see [GG22].

In the case f is a holomorphic, isometric embedding, the induced map $f^* : Q(Y) \to Q(X)$ is surjective instead of injective and we cannot apply Marković's result directly. Our workaround is to define *an umkehr* (backwards) map

$$f_! : Q(X) \to Q(Y)$$

going in the other direction. From the definition it will be clear that $f_!$ is continuous and norm preserving. The main technical difficulty is to show that in fact $f_!$ is $\mathbb{C}$-linear, this is done is Section 4. Afterwards we can apply Marković's theorem and then the proof roughly follows the proof of Royden's theorem given in [Mar03]. We will define the umkehr map in detail in Section 4 but the idea is as follows. A quadratic differential $q \in Q\mathcal{T}(X)$ in the cotangent space defines a geodesic g_q in $\mathcal{T}(\Sigma)$. The isometric embedding f maps geodesics in $\mathcal{T}(\Sigma)$ to geodesics in $\mathcal{T}(\Sigma')$. In particular the image $f(g_q)$ is a geodesic, which is generated by a quadratic differential q'. The umkehr map $f_!$ sends q to q'.

1.4. **Acknowledgements.** We would like to thank Benson Farb for many useful discussions and comments on an earlier draft. We also thank Howard Masur, Alex Wright and Dan Margalit for helpful discussions. Additionally, we thank the anonymous referee for useful suggestions that helped improve the clarity of the paper.

2. Preliminaries

Isometric embeddings and submersions. Let $(V, ||\cdot||_V), (W, ||\cdot||_W)$ be two finite- dimensional normed vector spaces and $T : V \to W$ be a linear map. We say T is an *isometric embedding* if $||v||_V = ||Tv||_W$ for all $v \in V$.

We say T is an *isometric submersion* if the image of the closed unit ball in V is the closed unit ball in W.

The dual of an isometric embedding is an isometric submersion and similarly, the dual of an isometric submersion is an isometric embedding. An isometric submersion can alternatively be characterized by the property

$$||v'||_W = \inf_{v \in V : Tv = v'} ||v||_V.$$

In a similar way isometric embeddings and submersion can also be defined for normed vector bundles. Here we recall that a normed vector bundle is a holomorphic vector bundle over a complex manifold with a function, which is C^1 outside the zero section and restricts to a norm on each fiber.

The Teichmüller metric. We start by recalling some basic facts about the Teichmüller metric. Fix g, n with $3g - 3 + n > 0$ and let $\mathcal{T}_{g,n}$ be the Teichmüller space of genus g surfaces with n marked points. If we need to specify a base point we write $\mathcal{T}_{g,n} = \mathcal{T}(\Sigma, z)$, where Σ is a compact topological surface and $z \subset \Sigma$ is a finite set of points. A point in $\mathcal{T}_{g,n}$ is determined by a compact Riemann surface X with a marking $\phi : \Sigma \to X$. The marked points on X are given by $\phi(z)$. If it is clear from the context we omit the marking ϕ and the marked points $\phi(z)$ and ambiguously write $X \in \mathcal{T}_{g,n}$. We always work with compact surfaces and marked points rather than punctured surfaces.

Let d be the Teichmüller metric on $\mathcal{T}_{g,n}$. Endowed with the metric d, Teichmüller space is a complete metric space such that through any two points there exists a unique geodesic.

The tangent space of $\mathcal{T}_{g,n}$ at $X \in \mathcal{T}_{g,n}$ can be identified with the space of all Beltrami forms, i.e., $(1, -1)$-forms μ on X with $||\mu||_\infty < \infty$, modulo infinitesimally trivial Beltrami forms. Dually the cotangent space $(T^*\mathcal{T}_{g,n})_X$ at X is identified with the space $Q(X)$ of integrable quadratic differentials on the punctured surface $X - \phi(z)$ or equivalently meromorphic quadratic differentials on X with at most simple poles at the marked points $\phi(z)$ and holomorphic elsewhere. If there is ambiguity about the marked points we write $Q(X, z)$ instead. Let $Q\mathcal{T}_{g,n}$ be the bundle of integrable quadratic differentials over $\mathcal{T}_{g,n}$. For $q \in Q(X)$, the *area norm* $||q||_{Q\mathcal{T}_{g,n}}$ is defined by

$$||q||_{Q\mathcal{T}_{g,n}} := \int_X |q|$$

and turns $Q\mathcal{T}_{g,n}$ into a normed vector bundle $(Q\mathcal{T}_{g,n}, ||\cdot||_{Q\mathcal{T}_{g,n}})$. The pairing between $T\mathcal{T}_{g,n}$ and $T^*\mathcal{T}_{g,n}$ is given by the pairing

$$\langle \mu, q\rangle \mapsto \int_X \mu q$$

between Beltrami forms and quadratic differentials. The infinitesimally trivial Beltrami forms μ are characterized by the condition

$$\langle \mu, q\rangle = 0 \text{ for all } q \in Q(X).$$

The dual of the area norm defines a norm $||\cdot||_{T\mathcal{T}_{g,n}}$ on the tangent space to $\mathcal{T}_{g,n}$ by

$$||\mu||_{T\mathcal{T}_{g,n}} = \sup_{\{\phi : ||\phi||_{Q\mathcal{T}_{g,n}}=1\}} \left| \int_X \mu\phi \right|.$$

The norm $||\mu||_{T\mathcal{T}_{g,n}}$ can be also computed as

$$||\mu||_{T\mathcal{T}_{g,n}} = \inf_{\mu'} ||\mu + \mu'||_\infty,$$

where μ' runs over all infinitesimally trivial Beltrami forms and $||\cdot||_\infty$ is the essential supremum. For a proof see [Hub16, Cor. 6.6.4.1].

The Teichmüller metric d is obtained by integrating $||\cdot||_{T\mathcal{T}_{g,n}}$, i.e., if $\gamma : [0,1] \to \mathcal{T}_{g,n}$ is a Teichmüller geodesic connecting $x, y \in \mathcal{T}_{g,n}$ then

$$d(x,y) = \int_0^1 ||\gamma'(t)||_{\mathcal{T}_{g,n}} dt.$$

The norm $||\cdot||_{T\mathcal{T}_{g,n}}$ is called the *infinitesimal Teichmüller metric*. Both norms $||\cdot||_{T\mathcal{T}_{g,n}}$ and $||\cdot||_{Q\mathcal{T}_{g,n}}$ are C^1 away from the zero section and strictly convex, see [Gar87, 9.3 Thm. 3 + Lemma 3]. Every Beltrami form μ of norm 1 can be represented uniquely as $\mu = \frac{\bar{q}}{|q|}$ for a quadratic differential q of area 1. Since the norm $||\cdot||_{Q\mathcal{T}_{g,n}}$ is strictly convex, the differential q can be characterized as the unique quadratic differential of norm 1 such that

$$\langle \mu, q\rangle = 1.$$

Teichmüller discs. Let $(X, q) \in Q\mathcal{T}_{g,n}$. The *Teichmüller disc* generated by q is the holomorphic map $g_q : \Delta \to \mathcal{T}_{g,n}$ given by $t \mapsto X_{\mu_t}$ where $\mu_t = t\dfrac{\bar{q}}{|q|}$ and X_{μ_t} is obtained by solving the Beltrami equation for μ_t. Results of Antonakoudis and Earle-Kra-Krushkal [Ant17, EKK94] show that every distance preserving map $\Delta \hookrightarrow \mathcal{T}_{g,n}$ for the Kobayashi metrics is a Teichmüller disc, up to complex conjugation. The complex tangent vector of g_q at the origin is $(g_{q,0})_*\partial_z = \dfrac{\bar{q}}{|q|}$. We refer to g_q as a complex geodesic. The restriction of g_q to any radial line is a real geodesic for the Teichmüller metric.

Totally geodesic submanifolds. A submanifold $M \subseteq \mathcal{T}_{g,n}$ is called *totally geodesic* if for all $x, y \in M$ the unique Teichmüller geodesic between x and y is contained in M. The *Teichmüller metric* d_M of M is the restriction of the Teichmüller metric of $\mathcal{T}_{g,n}$ to M. More generally if $M, N \subseteq \mathcal{T}_{g,n}$ are totally geodesic submanifolds and $M \subseteq N$, then we call M a totally geodesic submanifold of N. Every geodesic for the Teichmüller metric is the restriction of a holomorphic Teichmüller disc and thus any totally geodesic submanifold is a complex submanifold.

Definition 2.1. Let $M \subseteq \mathcal{T}_{g,n}$, $N \subseteq \mathcal{T}_{g',n'}$ be totally geodesic submanifolds of Teichmüller spaces. A map $f : M \to N$ is called an *isometric embedding* if it is distance-preserving for the Teichmüller metrics d_M and d_N, i.e.,

$$d_M(x, y) = d_N(f(x), f(y)) \text{ for all } x, y \in M.$$

In particular, an isometric embedding f sends geodesics to geodesics. By [Ant17, Thm. 4.3], any isometric embedding f is either holomorphic or anti-holomorphic. For the rest of this paper, we will assume that all isometric embeddings are holomorphic.

Proposition 2.2. *Let $f : M \to N$ be a holomorphic, isometric embedding of totally geodesic submanifolds of Teichmüller spaces. Then f is an immersion and $f(M) \subseteq N$ is a totally geodesic submanifold.*

Proof. For every tangent vector $v \in T_X M$ in the tangent space over $X \in M$ there exists a unique geodesic g_v through X with tangent vector v. The isometric embedding f sends g_v to a geodesic in N, which necessarily has non-zero tangent vector. Thus f is an immersion and $f(M)$ is a submanifold. It remains to see that $f(M)$ is totally geodesic. Let $x, y \in f(M)$ and choose preimages $a, b \in M$ with $f(a) = x, f(b) = y$. Since M is totally geodesic the unique geodesic $g_{a,b}$ through a and b is contained in M. The image $f(g_{a,b})$ is the unique geodesic through x and y. Here we used again that f sends geodesics to geodesics. □

Covering constructions. Let Σ, Σ' be topological surfaces of genus g and g', respectively, and $h : \Sigma' \to \Sigma$ a branched covering. Let $z \subseteq \Sigma$ a collection of marked points containing all branch points of h and set $z' = h^{-1}(z)$. There exists a holomorphic map

$$\begin{gathered} H : \mathcal{T}(\Sigma, z) \to \mathcal{T}(\Sigma', z'), \\ (X, \phi : \Sigma \to X, \phi(z)) \mapsto (X', \phi' : \Sigma' \to X', \phi'(z')), \end{gathered}$$

where X' is obtained by pulling back complex structures under the branched covering h.

Since Teichmüller maps pull back to Teichmüller maps, H is a holomorphic isometric embedding. We refer to H as a *totally marked covering construction.* A general covering construction is obtained from a totally marked covering constructions by forgetting marked points.

Definition 2.3. A *covering construction* is a holomorphic map $G : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v')$ fitting in a commutative diagram

$$\begin{array}{ccc} \mathcal{T}(\Sigma, z) & \xrightarrow{H} & \mathcal{T}(\Sigma', z') \\ \downarrow & & \downarrow \\ \mathcal{T}(\Sigma, u) & \xrightarrow[G]{} & \mathcal{T}(\Sigma', v) \end{array} \tag{1}$$

where the vertical maps are forgetful maps and H is a totally marked covering construction. In particular this requires $u \subseteq z, v \subseteq z'$.

Remark 2.4. The above definition might not appear as the most natural one, since depending on the choice of marked points $u \subseteq \Sigma, v \subseteq \Sigma'$ it is not obvious that G is well-defined and even if G is well-defined it might not be an isometric embedding. In the Appendix A we discuss sufficient and necessary conditions on the branched covering and the marked points $u \subseteq \Sigma, v \subseteq \Sigma'$ for the map G to be a well-defined isometric embedding. The results are not necessary for our proof of Theorem 1.1 and can be skipped during a first reading.

3. The geodesic bundle of totally geodesic submanifolds

The *geodesic bundle* $QM \subseteq Q\mathcal{T}_{g,n}$ over a totally geodesic submanifold is the set of all pairs (X, q) with $X \in M$ and q a quadratic differential generating a Teichmüller geodesic completely contained in M.

We start by showing that QM is a topological submanifold. Let $Q_1\mathcal{T}_{g,n}$ be the bundle of quadratic differentials with norm less than 1. The map

$$\begin{aligned} E :\, Q_1\mathcal{T}_{g,n} &\to \mathcal{T}_{g,n} \times \mathcal{T}_{g,n} \\ (X, q) &\mapsto (X, X_\mu), \end{aligned} \tag{2}$$

where $\mu = ||q|| \dfrac{\overline{q}}{|q|}$ and X_μ is obtained by solving the Beltrami equation for μ, is a homeomorphism by a result of Earle [Ear77]. Recall that complex geodesics are parametrized by $t \mapsto X_{\mu_t}$, where X_{μ_t} is the surface obtained by solving the Beltrami equation for $\mu_t = t\dfrac{\overline{q}}{|q|}$. Since M is totally geodesic we have

$$E(Q_1M) = M \times M, \tag{3}$$

where $Q_1M = Q_1\mathcal{T}_{g,n} \cap QM$. Therefore Q_1M is a topological submanifold and thus the same is true for QM.

Rees showed that E is not C^1 [Ree04, Thm. 1.], therefore it is *a priori* not clear that QM is even a real submanifold. The goal of this section is to show that in fact QM is a holomorphic subbundle of the cotangent space of $\mathcal{T}_{g,n}$.

Proposition 3.1. *Let $M \subseteq \mathcal{T}_{g,n}$ be a totally geodesic submanifold of dimension d. Then the geodesic bundle $QM \subseteq Q\mathcal{T}_{g,n}$ is a holomorphic subbundle of rank d.*

Proof. As a first step we show that QM is an analytic subvariety of $Q\mathcal{T}_{g,n}$. Let N be the projection of M to the moduli space $\mathcal{M}_{g,n}$ and similarly QN the projection of QM to $Q\mathcal{M}_{g,n}$. Wright [Wri20, Thm. 1.1] proved that N is an algebraic subvariety and QN is closed and $\mathrm{GL}(2, \mathbb{R})$-invariant. Thus, by Filip's theorem on algebraicity of $\mathrm{GL}(2, \mathbb{R})$-invariant subsets [Fil16, Thm 1.1], it follows that the intersection of QN with any stratum $Q\mathcal{M}_{g,n}(\mu)$ is an algebraic variety. We conclude that QN is closed and constructible, hence an algebraic variety. Now QM is an irreducible component of the preimage of QN in $\mathcal{T}_{g,n}$ and therefore an analytic subvariety.

It remains to show that QM is a linear subbundle of $Q\mathcal{T}_{g,n}$. Let T^*M be the cotangent bundle of M and

$$\iota : QM \to Q\mathcal{T}_{g,n|M} \to T^*M$$

be the composition of the inclusion with the pullback on cotangent bundles. In particular ι is holomorphic. We claim that ι is injective and proper.

To show the claim we will first show that ι is norm-preserving, where we endow T^*M with the quotient norm $|| \cdot ||_{T^*M}$.

Let $V = \ker(Q\mathcal{T}_{g,n|M} \to T^*M)$. Fix $q \in QM$ and let $q' = \iota(q)$. By definition of the quotient norm we have

$$||q'||_{T^*M} = \inf_{v \in V} ||q + v||.$$

Since $\dim(V) < \infty$, the infimum is achieved and since the norm $|| \cdot ||_{Q\mathcal{T}_{g,n}}$ on $Q\mathcal{T}_{g,n}$ is strictly convex the minimum is unique. It thus suffices to show that q is a critical point for the norm restricted to $q+V$. From now on assume $q \neq 0$. Royden [Roy71] showed that the norm is differentiable on $q+V$ and if we set $g(t) = ||q+tv||_{Q\mathcal{T}_{g,n}}$ for some $v \in V$, then

$$g'(0) = \mathrm{Re} \int_X v \frac{\overline{q}}{|q|}.$$

Thus q is a critical point if $\mathrm{Re} \int_X v \dfrac{\overline{q}}{|q|} = 0$ for all $v \in V$. Since V is a complex vector space this is equivalent to

$$\left\langle \tfrac{\overline{q}}{|q|}, v \right\rangle = \int_X v \tfrac{\overline{q}}{|q|} = 0 \text{ for all } v \in V.$$

Therefore $q \in Q\mathcal{T}_{g,n}$ is a critical point for the norm on $q+V$ if and only if $\frac{\overline{q}}{|q|} \in \mathrm{Ann}(V)$, where $\mathrm{Ann}(V)$ is the annihilator of V under the pairing $\langle \cdot, \cdot \rangle : T\mathcal{T}_{g,n} \times Q\mathcal{T}_{g,n} \to \mathbb{C}$.

To conclude the proof that ι is norm preserving it thus suffices to show $\frac{\overline{q}}{|q|} \in \mathrm{Ann}(V)$ for all $q \in QM$. Let $q \in QM$ and let $i : TM \hookrightarrow T\mathcal{T}_{g,n}$ be the inclusion of tangent bundles. Then the tangent vector $\frac{\overline{q}}{|q|}$ is contained in TM and ι is dual to i, thus

$$\langle \tfrac{\overline{q}}{|q|}, v \rangle = \langle i(\tfrac{\overline{q}}{|q|}), v \rangle = \langle \tfrac{\overline{q}}{|q|}, \iota(v) \rangle = 0 \text{ for all } v \in V.$$

Once we know that ι is norm preserving it follows that ι is injective, since if $q' = \iota(q)$ then q is the unique norm minimizer in $q+V$. To see that ι is also proper, take any sequence (X_n, q_n) in QM leaving any compact set. Then either X_n leaves any compact set in M or $||q_n||_{QM} \to \infty$ as $n \to \infty$. Since ι is norm-preserving it follows that $\iota(X_n, q_n)$ also leaves every compact set.

Thus the map $\iota : QM \to T^*M$ is holomorphic, injective and proper. Both QM and T^*M are topological submanifolds of the same dimension and thus ι is open by invariance of domain. Since M is connected, the same is true for T^*M. Thus ι is surjective and therefore a homeomorphism.

So far we have seen that ι is a holomorphic bijection. The next goal is to show that indeed ι is a biholomorphism. If it was already known that QM and T^*M are complex manifolds this would follow directly, since every holomorphic bijection between complex manifolds is biholomorphic. *A priori* QM has singularities; let $U, Z \subseteq QM$ be the smooth locus and singular locus, respectively. Then $\iota_{|U}$ is a holomorphic bijection of complex manifolds and thus a biholomorphism. Set $U' = \iota(U), Z' = \iota(Z)$. The proper mapping theorem implies that Z' is a proper analytic subvariety of T^*M and $\iota^{-1}_{|U'}$ is holomorphic. Thus by Riemann's removable singularity theorem the inverse ι^{-1} is holomorphic on T^*M. Note also that ι^{-1} is homogenous, since ι is. It follows that fiberwise $\iota^{-1}_X : T^*M_X \to (Q\mathcal{T}_{g,n})_X$ is homogenous and differentiable at the origin and therefore linear. □

The geodesic bundle QM is naturally a complex Finsler vector bundle by restricting the area norm from $Q\mathcal{T}_{g,n}$.

Corollary 3.2. *The normed vector bundle $(QM, ||\cdot||_{QM})$ is linearly isometric to the cotangent bundle $(T^*M, ||\cdot||_{T^*M})$ with the quotient norm and furthermore provides a splitting*

$$QM \oplus \mathcal{N}^*M = Q\mathcal{T}_{g,n|M}$$

as normed bundles. Here $\mathcal{N}^*M$ is the conormal bundle of M, i.e., the dual of the normal bundle, endowed with the restriction of the area norm on $Q\mathcal{T}_{g,n}$.

Proof. The isomorphism between QM and T^*M is given by the composition $\iota : QM \to Q\mathcal{T}_{g,n|M} \to T^*M$. For the second claim we need to show that QM and $\mathcal{N}^*M$ intersect only in the zero section. This follows from the description $\mathcal{N}^*M = \ker(Q\mathcal{T}_{g,n|M} \to T^*M)$. □

Remark 3.3 (Bilinearity of totally geodesic submanifolds). A surprising consequence of Proposition 3.1 is that the geodesic bundle QM of a totally geodesic submanifold is linear in two different ways. On one hand, the fibers of the projection $QM \to M$ are linear subspaces of the cotangent bundle and on the other hand the intersection of QM with a stratum coincides with a linear subspace in period coordinates of the stratum.

4. The geodesic umkehr map

Let $f : M \to N$ be a holomorphic isometric embedding of totally geodesic submanifolds of Teichmüller spaces. The derivative of f induces a map on tangent bundles $f_* : TM \to f^*TN$ and dually also on cotangent bundles $f^* : f^*T^*N \to T^*M$. By identifying the cotangent bundle with the geodesic bundle we obtain a map $f^* : f^*QN \to QM$.

Definition 4.1. The *geodesic umkehr (backwards) map*

$$f_! : QM \to f^*QN$$

is defined as follows. Let $(X, q) \in QM$ be a quadratic differential of area 1. Then q defines a Teichmüller geodesic g through X with tangent vector $\frac{\overline{q}}{|q|}$. Since f is an isometric embedding $f(g)$ is a geodesic through $f(X)$ with tangent vector $f_* \frac{\overline{q}}{|q|} = \frac{\overline{q'}}{|q'|}$ for a unique quadratic differential q' of unit area. Define

$$f_!(X, q) = (f(X), q').$$

Extend $f_!$ to a homogeneous function on all of QM by defining

$$f_!(X, q) := ||q||_{QM} f_!(X, \tfrac{1}{||q||_{QM}} q)$$

for non-zero quadratic differentials q and zero otherwise. In particular, with this definition

$$f_* \frac{\overline{q}}{|q|} = \frac{\overline{f_! q}}{|f_! q|}, \text{ if } q \neq 0. \tag{4}$$

Remark 4.2. The umkehr map is most natural in the case of a covering construction

$$f : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$$

arising from a branched covering

$$h : \Sigma' \to \Sigma$$

of degree d. For $X \in \mathcal{T}(\Sigma), Y = f(X) \in \mathcal{T}(\Sigma')$, the dual of the derivative of f induces a pullback $f^* : Q(Y) \to Q(X)$, which is given by the trace map

$$\mathrm{tr_h} : Q(Y) \to Q(X).$$

Recall that the trace map is defined by

$$\mathrm{tr}_h(q)(x) = \sum_{y \in h^{-1}(x)} q(y),$$

if $x \in X$ is not a branched point and extended continously to branch points. The umkehr map $f_! : Q(\Sigma) \to Q(\Sigma')$ on the other hand is given by pullback of differentials $f_! = \frac{1}{d} h^*$. Note that in particular $f_!$ is linear, isometric for the area norm on quadratic differentials and a right inverse to f^*. One of our main technical theorems is that the same conclusions hold for any isometric embedding of totally geodesic submanifolds of Teichmüller spaces.

Observe that $f_!$ fits into the commutative diagram

$$\begin{array}{ccc} Q_1 M & \xrightarrow{f_!} & f^* Q_1 N \\ {\scriptstyle E_M}\big\downarrow & & \big\downarrow{\scriptstyle f^* E_N} \\ M \times M & \xrightarrow[f \times f]{} & N \times N \end{array} \tag{5}$$

where $E_M : Q_1 M \to M \times M$ and $E_N : Q_1 N \to N \times N$ are the homeomorphisms obtained by restricting E from eq. (2) to M and N, respectively. It follows from the above diagram that

$$f_! = (f^* E_N)^{-1} \circ (f \times f) \circ E_M$$

is a composition of continuous maps and thus continuous on $Q_1 M$, and thus by homogeneity on all QM. The main result of this section is that $f_!$ is indeed much better behaved.

Proposition 4.3 (Regularity of $f_!$)**.** *Let $M \subset \mathcal{T}_{g,n}$, $N \subseteq \mathcal{T}_{g',n'}$ totally geodesic submanifolds and let $f : M \to N$ be a holomorphic isometric embedding. Then the umkehr map*

$$f_! : QM \to f^* QN$$

has the following properties.

1. *The map $f_!$ is a holomorphic, $\mathbb{C}$-linear embedding of vector bundles,*
2. *an isometry for the area norms on QM and QN and*
3. *a right inverse of $f^* : f^* QN \to QM$.*

Furthermore the formation of $f_!$ is functorial for isometric embeddings of totally geodesic submanifolds.

Proof. It suffices to deal with the case $N = f(M)$. Note that $f_! QM = QN$ and $f^* : f^* QN \to QM$ is holomorphic. We will first show that $f^* \circ f_! = \mathrm{id}$, i.e., $f_!$ is a right inverse of f^*.

Recall that the pairing between Beltrami forms in TM and quadratic differentials in QM is given by $(\mu, q) = \int_X \mu q$. Also recall that every Beltrami form can be represented uniquely in its equivalence class as $\mu = ||\mu||_\infty \frac{\bar{\phi}}{|\phi|}$. Let $(X, q) \in QM$ and $k = ||q||_{QM}$. Since the Teichmüller metric is dual to the area norm on quadratic differentials we have

$$k = ||k \frac{\bar{q}}{|q|}||_{TM} = k \sup_{||\phi||=1} \left| \int_X \frac{\bar{q}}{|q|} \phi \right|$$

and as the norm $|| \cdot ||_{TM}$ is strictly convex, there is a unique quadratic differential ϕ of norm 1 such that

$$k = ||k \frac{\bar{q}}{|q|}||_{TM} = \int_X k \frac{\bar{q}}{|q|} \phi.$$

In fact $\phi = \dfrac{q}{||q||_{QM}}$, since

$$\int_X k\frac{\overline{q}}{|q|} \cdot \frac{q}{||q||_{QM}} = \frac{k}{||q||_{QM}} \int_X |q| = k$$

First observe that $f^* f_! q \neq 0$ unless $q = 0$. Indeed, recall that by Equation (4)

$$f_* \frac{\overline{q}}{|q|} = \frac{\overline{f_! q}}{|f_! q|}.$$

Then, the claim follows from

$$\int_X \frac{\overline{q}}{|q|} (f^* f_! q) = \int_{f(X)} \left(f_* \frac{\overline{q}}{|q|} \right) f_! q = \int_{f(X)} \frac{\overline{f_! q}}{|f_! q|} f_! q = ||f_! q||_{QN} = ||q||_{QM}.$$

Furthermore

$$||f^* f_! q||_{QM} = \inf_{\{q' : f^* q' = f^* f_! q\}} ||q'||_{QN} \leq ||f_! q||_{QN} = ||q||_{QM},$$

where the first equality follows since f^* is an isometric submersion.

For $q \neq 0$ we now compute

$$\begin{aligned} \int_X \frac{\overline{q}}{|q|} \cdot \frac{f^* f_! q}{||f^* f_! q||_{QM}} &= \int_{f(X)} \left(f_* \frac{\overline{q}}{|q|} \right) \cdot \frac{f_! q}{||f^* f_! q||_{QM}} \\ &= \int_{f(X)} \frac{\overline{f_! q}}{|f_! q|} \cdot \frac{f_! q}{||f^* f_! q||_{QM}} \\ &= \frac{||f_! q||_{QN}}{||f^* f_! q||_{QM}} \geq 1. \end{aligned}$$

On the other hand

$$\left| \int_X \frac{\overline{q}}{|q|} \cdot \frac{f^* f_! q}{||f^* f_! q||_{QM}} \right| \leq ||\frac{\overline{q}}{|q|}||_\infty \int_X \frac{|f^* f_! q|}{||f^* f_! q||_{QM}} = 1$$

we conclude $f^* f_! q = q$. This finishes the proof of (1).

By definition $f_! : QM \to f^* QN$ is norm-preserving, hence injective and proper. Additionally

$$\dim QN = 2 \dim N = 2 \dim M = \dim QM,$$

which follows from eq. (3). Now both QM and QN are connected, complex manifolds of the same dimension, thus by invariance of domain $f_! : QM \to f^* QN$ and $f^* : f^* QN \to QM$ are bijective and inverses of each other. Since f^* is linear the same is true for $f_!$.

Finally, the functoriality of $f_!$ follows from eq. (5) and the functoriality of f_*.

□

Remark 4.4. Using eq. (5) we can define the umkehr map more generally for any holomorphic map $f : M \to N$ between totally geodesic submanifolds and from the diagram it follows that $f_!$ is functorial for holomorphic maps. In case $f : M \to N$ is an isometric submersion of totally geodesic submanifolds we can also give a geometric description for $f_!$ as follows. Since f_* is an isometric submersion the pullback map

$$f^* : f^* QN \to QM$$

is an isometric immersion and we have the relation

$$f_! f^* q = q.$$

To see this note that it suffices to show

$$f_* \left(\frac{\overline{f^* q}}{|f^* q|} \right) = \frac{\overline{q}}{|q|}.$$

Suppose that $||q||_{QN} = 1$. Then

$$\int_{f(X)} f_* \left(\frac{\overline{f^* q}}{|f^* q|} \right) q = ||f^* q||_{QM} = ||q||_{QN} = 1$$

and since f_* is an isometric submersion

$$||f_* \left(\frac{\overline{f^* q}}{|f^* q|} \right) ||_{TN} \leq || \frac{\overline{f^* q}}{|f^* q|} ||_{TM} = 1.$$

We thus conclude $f_! f^* q = q$.

5. Isometric embeddings are geometric

In this section, using ideas from Gekhtman and Greenfield [GG22, Section 3], we complete the proof of Theorem 1.1. The link between the linearity of the geodesic umkehr map $f_!$ and a covering construction is provided by the following theorem of [GG22, Theorem 1.3], which relives heavily on ideas coming from a paper of Marković [Mar03].

Theorem 5.1 ([GG22, Theorem 1.3])**.** *Let X and Y be compact marked Riemann surfaces. Assume $2g(X)+n(X) \geq 5$, where $n(X)$ is the number of marked points. Let $T : Q(X) \to Q(Y)$ be a $\mathbb{C}$-linear isometric embedding. Then there is a holomorphic map $h : Y \to X$, and a constant $c \in \mathbb{C}, |c| = 1$ such that $T = c \cdot \dfrac{h^*}{\deg(h)}$.*

Remark 5.2. In fact, the proof of Theorem 5.1 gives an explicit description of the map h as follows. Let $\Psi_X : X \to \mathbb{P}Q(X)^*$ and $\Psi_Y : Y \to \mathbb{P}Q(Y)^*$ be the bicanonical embeddings for X and Y.

Then h fits into the diagram:

$$\begin{tikzcd} \mathbb{P}Q(Y)^* \arrow[r, "T^*"] & \mathbb{P}Q(X)^* \\ Y \arrow[u, "\Psi_Y"] \arrow[r, "h"] & X \arrow[u, "\Psi_X"'] \end{tikzcd}$$

We emphasize here that $Q(X)$ denotes the space of meromorphic quadratic differentials with at most simple poles at the marked points of X, so that Ψ_X might differ from the usual bicanonical embedding in algebraic geometry. The condition $2g(X) + n(X) \geq 5$ guarantees that Ψ_X is an embedding.

We now apply Theorem 5.1 in the situation where $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ is a holomorphic isometric embedding. Let $\mathcal{C}_{g,n}$ and $\mathcal{C}_{g',n'}$ be the universal curves over $\mathcal{T}_{g,n}$ and $\mathcal{T}_{g',n'}$. Applying Theorem 5.1 fiberwise to the umkehr map $f_! : Q\mathcal{T}_{g,n} \to f^* Q\mathcal{T}_{g',n'}$, gives a map $H : f^*\mathcal{C}_{g',n'} \to \mathcal{C}_{g,n}$, which fits into the commutative diagram,

$$\begin{tikzcd} f^*\mathcal{C}_{g',n'} \arrow[r, "H"] \arrow[d] & \mathcal{C}_{g,n} \arrow[d] \\ f(\mathcal{T}_{g,n}) & \mathcal{T}_{g,n} \arrow[l, "f"'] \end{tikzcd}$$

As we will see below, the regularity of H plays a key role in the proof of Theorem 1.1. The following shows that H is holomorphic.

Lemma 5.3. *Suppose $2g + n \geq 5$. Then the map $H : f^*\mathcal{C}_{g',n'} \to \mathcal{C}_{g,n}$ is holomorphic.*

Proof. Consider the following commutative diagram,

$$\begin{tikzcd} \mathcal{C}_{g,n} \arrow[r, "\Psi"] \arrow[dr] & \mathbb{P}Q\mathcal{T}^*_{g,n} \arrow[d] & f^*\mathbb{P}Q\mathcal{T}^*_{g',n'} \arrow[l, "f^*_!"'] \arrow[d] & f^*\mathcal{C}_{g',n'} \arrow[l, "\Phi"'] \arrow[dl] \\ & \mathcal{T}_{g,n} \arrow[r, "f"] & f(\mathcal{T}_{g,n}) & \end{tikzcd}$$

The maps Ψ and Φ are the respective fiberwise bicanonical embeddings, which are biholomorphic onto their image. Since H is given by

$$H = \Psi^{-1} \circ f^*_! \circ \Phi,$$

the claim follows. $\square$

Finishing the proof of Theorem 1.1. For the rest of the proof we identify $\mathcal{T}_{g,n} = \mathcal{T}(\Sigma, u)$ with its image $f(\mathcal{T}_{g,n}) \subset \mathcal{T}_{g',n'} = \mathcal{T}(\Sigma', v)$. By Lemma 5.3, there exists a holomorphic family of branched coverings

$$\begin{tikzcd} f^*\mathcal{C}_{g',n'} \arrow[d, "H"] \\ \mathcal{C}_{g,n} \arrow[d] \\ \mathcal{T}_{g,n}. \end{tikzcd}$$

For a marked surface $(X, u, \phi : \Sigma \to X) \in \mathcal{T}_{g,n} = \mathcal{T}(\Sigma, u)$ we let $h_X : f(X) \to X$ be map induced by restricting H to a fiber. The ramification multiplicity $(X, x) \to \operatorname{mult}_x h_X$ is an upper semicontinuous function on $f^*\mathcal{C}_{g',n'}$. Indeed, this can be seen as follows: $\pi_1 : f^*\mathcal{C}_{g',n'} \to \mathcal{T}_{g,n}$ is a holomorphic submersion, and the same is true for $\pi_2 : \mathcal{C}_{g,n} \to \mathcal{T}_{g,n}$. Thus locally around $(X_0, x_0) \in f^*\mathcal{C}_{g',n'}$ we can reduce to

$$H : \Delta^{3g-3} \times \Delta \to \Delta^{3g-3} \times \Delta, H(b, x) = (b, g(b, x)).$$

for some holomorphic function g such that $g(0, x) = x^{\operatorname{mult}_{x_0} h_{X_0}}$. Then up to shrinking the polydisks, it follows from the *Weierstrass preparation theorem* that

$$g(b, x) = (x^N + a_{N-1}(b)x^{N-1} + \ldots + a_1(b)x + a_0(b))u(b, x),$$

where $N = \operatorname{mult}_{x_0} h_{X_0}$, a_i are holomorphic, vanish at 0 and u is holomorphic and $u(b, x) \neq 0$. As $\partial_x^N g(0, x) \neq 0$ we have, by shrinking if needed, that $\partial_x^N g(b, x) \neq 0$ for all (b, x). Hence, the multiplicity can only go down along closed analytic subvarieties, and upper semicontinuity follows. In particular, H is a local biholomorphism if x_0 is not a ramification point of h_{X_0}. Thus, by compactness of the fibers it follows that $\deg h_X$ is *constant.*

Let

$$M_k := \{(X,x) : \operatorname{mult}_x h_X \geq k\} \subseteq f^*\mathcal{C}_{g',n'}.$$

M_k is a closed analytic set, $M_k \subseteq M_{k-1}$ and $M_k = \emptyset$ for k big enough. Let $\{Y_\ell\}$ be the irreducible components of M_2 of maximal dimension, and denote by W the union of components of M_2 of dimension strictly less than $3g-3+n$ (note that this can be all of M_2). Let $Z_\ell \subsetneq Y_\ell$ be the singular locus of Y_ℓ, and $V_\ell = Y_\ell \cap M_{k_\ell}$, where k_ℓ is the first k so that $Y_\ell \not\subseteq M_k$. Then,

$$Z = \cup_\ell Z_\ell \cup_\ell V_\ell \cup W$$

is a closed analytic set with $\dim(Z) < 3g-3+n$. By construction, $Y_\ell - Z$ is a connected analytic variety where the map $(X,x) \mapsto \operatorname{mult}_x h_X$ is constant. Furthermore, the Weierstrass preparation argument also shows that $Y_\ell - Z$ intersects the fibers of π_1 transversely. Let $U = \mathcal{T}_{g,n} \setminus \pi_1(Z)$. Then $\pi_1(Z)$ is an analytic subset of $\mathcal{T}_{g,n}$ by the proper mapping theorem, see for example [Fis06, 1.19, Semiproper mapping theorem] for a reference. Since $\dim \pi_1(Z) < 3g-3+n$, the open set U is the complement of a proper closed analytic subvariety and hence U is a connected dense open set. Additionally, we claim that the *ramification profile*, i.e., the number of ramification points and ramification multiplicities is constant on U.

To see the claim, we distinguish two cases. If $Z = M_2$, then over U the branched covering h_X is in fact unramified. On the other hand, if $Z \subsetneq M_2$, then

$$\dim Y_\ell = \dim U$$

and the forgetful map $\pi_{1|Y_\ell - Z} : Y_\ell - Z \to U$ is a proper, local biholomorphic map (here we used that $Y_\ell - Z$ intersects the fibers of π_1 transversely). Hence π_1 restricted to $Y_\ell - Z$ is a covering map and has constant degree. This implies that the ramification profile is constant. For each (X,ϕ,u), let $z(X)$ be the union of u and all branch points of h_X. Similarly, let $z'(X)$ be the union of all marked points v on $f(X)$ and all ramification points. By removing z and z' and varying X we obtain a holomorphic family of covering maps

$$f(X) - z'(X) \to X - z(X).$$

By passing to the universal cover $\widehat{U}$ of U we obtain marked families and furthermore, the family of coverings can be (smoothly) trivialized by covering space theory. Summarizing, there is a diagram (*a priori* not commutative)

$$\begin{array}{ccccc}
\widehat{U} & \longrightarrow & \mathcal{T}(\Sigma, z) & \xrightarrow{g} & \mathcal{T}(\Sigma', z') \\
\downarrow & & \downarrow \pi & & \downarrow \\
U & \longrightarrow & \mathcal{T}(\Sigma, u) & \xrightarrow[f]{} & \mathcal{T}(\Sigma', v).
\end{array}$$

The first diagram is commutative and the second one is commutative at least over $\pi^{-1}(U)$, which is dense in $\mathcal{T}(\Sigma, z)$. Since all maps involved are holomorphic we conclude that the whole diagram commutes. By construction, g is a totally marked covering construction and thus f is a covering construction by definition.

□

5.1. **Local isometric embeddings of moduli spaces.** The goal of this subsection is to prove Corollary 1.4. Let $\Gamma \subseteq \operatorname{Mod}(\Sigma)$ be a subgroup of the mapping class group. The mapping class group $\operatorname{Mod}(\Sigma)$ acts by isometries on $\mathcal{T}(\Sigma)$ and we endow the quotient $\mathcal{T}(\Sigma)/\Gamma$ with the quotient metric. Outside the orbifold locus (curves with extra automorphisms), the quotient map $\mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)/\Gamma$ is a local isometry.

For any subgroup $\Gamma \subseteq \mathrm{Mod}(\Sigma)$, the *coarse moduli space*

$$M(\Gamma) := \mathcal{T}(\Sigma)/\Gamma$$

is the quotient in the category of complex-analytic spaces and

$$\mathcal{M}(\Gamma) := [\mathcal{T}(\Sigma)/\Gamma]$$

the associated orbifold.

A holomorphic map $M(\Gamma) \to M(\Gamma')$ of coarse moduli spaces is a holomorphic map of the underlying complex-analytic spaces. On the other hand, a holomorphic map

$$f : \mathcal{M}(\Gamma) \to \mathcal{M}(\Gamma')$$

of orbifolds is the same as a group homomorphism $\rho : \Gamma \to \Gamma'$ and a ρ-equivariant, holomorphic map $\tilde{f} : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$. A holomorphic map of orbifolds or coarse moduli spaces is called a *local isometric embedding* if there exists a dense open set $U \subseteq \mathcal{T}(\Sigma)$, contained in the complement of the orbifold locus, such that the restriction $\tilde{f}_{|U}$ is a local isometry for the quotient metric. This definition is justified by the following lemma.

Lemma 5.4. *Suppose an isometric embedding $\tilde{f} : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$ descends to a holomorphic map $M(\Gamma) \to M(\Gamma')$ of coarse moduli spaces for subgroups $\Gamma \subseteq \mathrm{Mod}(\Sigma)$, $\Gamma' \subseteq \mathrm{Mod}(\Sigma')$. Then f is a local isometric embedding. On the other hand, let $f : \mathcal{M}(\Gamma) \to \mathcal{M}(\Gamma')$ be a local isometric embedding of orbifolds with ρ-equivariant lift $\tilde{f} : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma')$. Then $\tilde{f}$ is an isometric embedding.*

Proof. If $\tilde{f}$ is an isometric embedding, so is the restriction to the complement of the orbifold locus in $\mathcal{T}(\Sigma)$. The action of Γ and Γ' is properly discontinuous and free on the complement of the orbifold locus. Suppose the image of $\tilde{f}$ is not contained in the orbifold locus of $\mathcal{T}(\Sigma')$. Then, after further restricting to the complement of the preimage of the orbifold locus in $\mathcal{T}(\Sigma')$, we can pass to the quotients $M(\Gamma)$ and $M(\Gamma')$, and obtain a local isometric embedding on some dense open subset. Now suppose the image of $\tilde{f}$ is completely contained in the orbifold locus. The image $\tilde{f}(\mathcal{T}(\Sigma))$ is irreducible and thus the stabilizer of a point for the mapping class group action is constant on a dense open set. Since the action of $\mathrm{Mod}(\Sigma')$ is properly discontinuous, there is a proper analytic subset of $\tilde{f}(\mathcal{T}(\Sigma))$, outside of which the projection to $M(\Gamma')$ is a local isometry. Hence, the same argument as in the first case gives the claim.

On the other hand, suppose that $f : \mathcal{M}(\Gamma) \to \mathcal{M}(\Gamma')$ is a local isometric embedding with lift $\tilde{f}$. We can find an open dense set $U \subseteq \mathcal{T}(\Sigma)$ such that $\tilde{f}$ restricts to a local isometric embedding

$$\tilde{f}_{|U} : U \to \mathcal{T}(\Sigma'),$$

In particular, the induced map $\tilde{f}_* : TU \to T\mathcal{T}(\Sigma' x)$ on the tangent bundles is norm-preserving and thus on all of $T\mathcal{T}(\Sigma)$ by continuity of the norm. Any norm-preserving holomorphic map of Teichmüller spaces is an isometric embedding, which can be seen by restricting to Teichmüller geodesics and using [EKK94, Thm. 5].

□

We have now all the ingredients to prove Corollary 1.4. The main idea is to reduce the proof to Theorem 1.1 by lifting local isometric embeddings to isometric embeddings of Teichmüller spaces.

Proof of Corollary 1.4. Since $\dim \mathcal{T}_{g,n} \geq 2$, Theorem 1.1 and Corollary 1.2 imply that the isometric embedding $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ is a covering construction. Let $\operatorname{Mod}(g,n) = \operatorname{Mod}(\Sigma, u)$ be the mapping class group of n-marked genus g surfaces. Given a branched covering $h : \Sigma' \to \Sigma$, let $\operatorname{Mod}_h \subseteq \operatorname{Mod}(g,n)$ be the finite-index subgroup of mapping classes that lift under the branched covering h. The covering construction then descends to a local isometric embedding

$$\widetilde{f} : \widetilde{M}_{g,n} = \mathcal{T}_{g,n}/\operatorname{Mod}_h \to M_{g',n'}$$

of coarse moduli spaces by Lemma 5.4. Note that there is not necessarily an induced group homomorphism $\operatorname{Mod}_h \to \operatorname{Mod}(g',n')$, thus we cannot guarantee a map between orbifolds.

Conversely, let

$$\widetilde{\mathcal{M}}_{g,n} = [\mathcal{T}_{g,n}/\Gamma],\ \widetilde{\mathcal{M}}_{g',n'} = [\mathcal{T}_{g',n'}/\Gamma']$$

be finite orbifold covers of $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,'n,'}$, respectively, for some finite index subgroups

$$\Gamma \subseteq \operatorname{Mod}(g,n),\ \Gamma' \subseteq \operatorname{Mod}(g',n').$$

Let

$$\widetilde{f} : \widetilde{\mathcal{M}}_{g,n} \to \widetilde{\mathcal{M}}_{g',n'}$$

be a holomorphic, local isometric embedding of orbifolds. In particular, $\widetilde{f}$ is induced by a ρ-equivariant norm-preserving map of Teichmüller spaces $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',m'}$, where $\rho : \Gamma \to \Gamma'$ is some group homomorphism. Then, again by Lemma 5.4, f is an isometric embedding and thus a covering construction by Theorem 1.1. Hence f is obtained by pulling back complex structures along a branched covering $h : \Sigma \to \Sigma'$ of topological surfaces. Pre- and post composing with homeomorphisms isotopic to the identity changes the branched covering h but not the induced map f on Teichmüller spaces. For fixed tuples (g, g', n, n') the mapping class groups $\operatorname{Mod}(g,n)$ and $\operatorname{Mod}(g',n')$ act on

$$\operatorname{Cov} := \{\phi : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'} \,|\, \phi \text{ is a covering construction}\}$$

by pre- and post-composition, respectively. Both actions by $\operatorname{Mod}(g,n)$ and $\operatorname{Mod}(g',n')$ leave the ramification profile and the monodromy representation of h invariant. If a covering construction ϕ descends to a map

$$\widetilde{\phi} : \widetilde{\mathcal{M}}_{g,n} \to \widetilde{\mathcal{M}}_{g',n'},$$

then $\gamma'\phi\gamma$ for $\gamma \in \Gamma, \gamma' \in \Gamma'$ descends to the same map $\widetilde{\phi}$. In particular there are at most

$$[\operatorname{Mod}(g',n') : \Gamma'] \cdot [\operatorname{Mod}(g,n) : \Gamma]$$

coverings constructions

$$\widetilde{\phi} : \widetilde{\mathcal{M}}_{g,n} \to \widetilde{\mathcal{M}}_{g',n'},$$

with a fixed ramification profile and fixed monodromy representation. Since there are only finitely many choices for both ramification and monodromy representation, there are only finitely many coverings constructions from $\widetilde{\mathcal{M}}_{g,n}$ to $\widetilde{\mathcal{M}}_{g',n'}$.

$\square$

Appendix A. Covering constructions

In Definition 2.3, covering constructions

$$G : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$$

associated to a branched covering

$$h : \Sigma' \to \Sigma$$

of topological surfaces were defined. But given a branched covering $h : \Sigma' \to \Sigma$ it is not always true that there exists a well-defined covering construction arising from h. The goal of this appendix is to discuss necessary and sufficient conditions for a branched covering to give rise to an isometric covering construction. Given h, one can always choose the branched points $u \subseteq \Sigma, v \subseteq \Sigma'$ such that G is well-defined. For example, if u is the set of branch points of h and $v = h^{-1}(u)$, then G is well-defined and an isometric embedding. But for other choices of u and v the map G might not be well-defined or not an isometric embedding. In the rest of the appendix we discuss in detail when G is well-defined and if so, when G is an isometric embedding.

A covering construction $G : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$ is uniquely determined by the diagram eq. (1), i.e., by the data of a *branching tuple* (h, u, v, z) consisting of

- a branched covering $h : \Sigma' \to \Sigma$,
- finite sets $u \subseteq z \subseteq \Sigma$ such that z contains all branch points of h and
- a finite set $v \subseteq h^{-1}(z) \subseteq \Sigma'$.

Definition A.1. A branching tuple (h, u, v, z) associated to a branched covering $h : \Sigma' \to \Sigma$ is *realizable* if there exists a holomorphic map G fitting into a diagram (1) where H is a totally marked covering construction. Similarly, the branching tuple (h, u, v, z) is *realizable by an isometric covering construction* if G is an isometric embedding.

The basic existence result for covering constructions is that a branching tuple can be realized by an isometric embedding, if every branch point is marked, the image of every marked point is marked and the preimage of a marked point is either a marked point of a ramification point. The precise statement is as follows.

Proposition A.2. *Suppose (h, u, v, z) is a branching tuple associated to a branched covering $h : \Sigma' \to \Sigma$ such that*

$$h^{-1}(u) = v \cup R(h),$$

where $R(h) \subseteq \Sigma'$ is the set of ramification points. Then (h, u, v, z) is realizable by an isometric embedding.

Proof. See [LS14, Prop. 4.4] for the case $u = B(h)$, where $B(h)$ is the set of branch points. The proof of the general case is identical, since every quadratic differential with at most simple poles at u can be lifted to a quadratic differential with at most simple poles at $h^{-1}(u) - R(h)$ and thus also Teichmüller maps can be lifted. □

The goal of this appendix is to show that, except for some low genus cases, the condition

$$h^{-1}(u) = v \cup R(h)$$

is a necessary and sufficient condition for a branching tuple to be realized by an isometric embedding.

We first address the case where (h,z,u,v) is already assumed to be realizable and discuss necessary and sufficient conditions for when the resulting covering construction G is an isometric embedding.

Proposition A.3. *Let $G : \mathcal{T}(\Sigma,u) \to \mathcal{T}(\Sigma',v)$ be a covering construction induced by branched covering $h : \Sigma' \to \Sigma$ and assume $\mathcal{T}(\Sigma,u) \not\simeq \mathcal{T}_{1,1}$. Then G is an isometric embedding if and only if*

$$h^{-1}(u) \subseteq v \cup R(h),$$

where $R(h) \subseteq \Sigma'$ is the set of ramification points of h.

Proof. Suppose G is an isometric embedding. In particular G fits into a diagram

$$\begin{tikzcd} \mathcal{T}(\Sigma,z) \arrow[r, "H"] \arrow[d] & \mathcal{T}(\Sigma',h^{-1}(z)) \arrow[d] \\ \mathcal{T}(\Sigma,u) \arrow[r, "G"'] & \mathcal{T}(\Sigma',v) \end{tikzcd} \tag{6}$$

where H is a totally marked covering construction. Given $X \in \mathcal{T}(\Sigma,z)$ and $X' = H(X) \in \mathcal{T}(\Sigma',h^{-1}(z))$, there are induced holomorphic branched coverings

$$h_X : X' \to X$$

The geodesic umkehr map $H_!$ is given by pullback of quadratic differentials

$$h_X^* : Q(X,z) \to Q(X',z').$$

All four maps in the diagram (6) are either isometric immersions or isometric submersions, and thus, by the functoriality of the umkehr map, we see that $G_!$ is also given by pullback h_X^*. In particular, there exists an induced map

$$(h_X^*)_{|Q(X,u)} : Q(X,u) \to Q(X',v).$$

Let $p \in u$ be a marked point. Since $(g,n) \neq (1,1)$, there exists a quadratic differential q on X with a simple pole at p (and potentially more simple poles). The pullback differential $h_X^*(q)$ has a simple pole at every preimage of p, unless the preimage is a ramification point. Since simple poles have to be marked points, we conclude that v contains $h^{-1}(u) \setminus R(h)$.

To see sufficiency, note that a Teichmüller map lifts to the branched covering, as long as a quadratic differential with simple poles at u pulls back to a quadratic differential with at most simple poles at v, which is guaranteed by $h^{-1}(u) \subseteq v \cup R(h)$.

□

Now we address the same question, but without assuming (h,u,v,z) is already realizable.

Proposition A.4. *Let $(\Sigma,u),(\Sigma',v)$ be topological surfaces with $g(\Sigma) \geq 1$ or $g(\Sigma') \geq 2$ and $(g(\Sigma),|u|) \neq (1,1)$. Furthermore, let (h,u,v,z) be a branching tuple. Then (h,u,v,z) is realized by an isometric covering construction $G : \mathcal{T}(\Sigma,u) \to \mathcal{T}(\Sigma',v)$ if and only if*

$$h^{-1}(u) = v \cup R(h),$$

where $R(h) \subseteq \Sigma'$ is the set of ramification points of h.

Proof. The if direction is part of Proposition A.2.

To prove the converse, assume that G is an isometric covering construction for some branching tuple (h,z,u,v). By Proposition A.3 it follows

$$h^{-1}(u) \subseteq v \cup R(h).$$

We will first argue that $h(v) \subseteq u$, i.e., every marked point of X' lies over a marked point of X. The coderivative map of G can be computed as the trace map

$$\mathrm{tr}_{h_X} : Q(X', v) \to Q(X, u),$$

by considering the diagram eq. (6). Assume $p \in v \setminus (R(h) \cup h^{-1}(u))$.

If there exists a quadratic differential $q' \in Q(X', v)$ with a simple pole at p but no other poles in the fiber $h_X^{-1}(h(p))$, then $\mathrm{tr}_X(q')$ has a simple pole at $h(p)$ and thus $h(p)$ needs to be marked as well. If $g(X') \geq 2$ there exists a quadratic differential with a simple pole at p and no other poles by Riemann-Roch.

If $g(X') = 1$ and $g(X) = 1$, then h_X is unramified by Riemann-Hurwitz. Since u has at least two points the same is true for $h^{-1}(u)$. Furthermore, p is not contained in $h^{-1}(u)$, and thus there exists a quadratic differential with a simple pole at p such that the trace has a simple pole at $h(p)$.

So far we have shown

$$h^{-1}(u) \subseteq v \cup R(h) \subseteq h^{-1}(u) \cup R(h).$$

Next, we argue that u contains all the branch points $B(h)$ and thus

$$h^{-1}(u) = v \cup R(h).$$

The map G fits in a commutative diagram

$$\begin{tikzcd}
\mathcal{T}(\Sigma, u \cup B(h)) \arrow[r, "K"] \arrow[d, "\pi"'] & \mathcal{T}(\Sigma', h^{-1}(u) \cup R(h)) \arrow[d, "\pi'"] \\
\mathcal{T}(\Sigma, u) \arrow[r, "G"'] & \mathcal{T}(\Sigma', v)
\end{tikzcd}$$

Here, K is a totally marked covering construction and hence an isometric embedding. The composition $\pi' \circ K$ is the composition of a totally marked covering construction followed by a forgetful map which forgets some ramification points. Hence $\pi' \circ K$ is realizable by an isometric embedding by Proposition A.2. It folllows from the diagram that π is injective, which forces $B(h) \subseteq u$.

□

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